Integrate $\boldsymbol{I = \int e^{ax}\cos(bx+c)\,dx}$, $\boldsymbol{J = \int e^{ax} sin(bx+c)\,dx}$

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Method 1 (Integration by parts)

For $I = \int e^{ax}\cos(bx+c)\,dx$

Let $u = \cos(bx+c)$, $du = -b\sin\ (bx+c)\,dx$, $dv = e^{ax}dx$, $v = \frac{1}{a}e^{ax}$

$\therefore I = \frac{1}{a}e^{ax}\cos(bx+c) - \int\left(\frac{1}{a}e^{ax}\right)[-b\sin(bx+c)\,dx] = \frac{1}{a}e^{ax}\cos(bx+c) + \frac{b}{a}\int e^{ax}\sin(bx+c)\,dx$

Since $J = \int e^{ax}\sin(bx+c)\,dx$

$I - \frac{b}{a}J = \frac{1}{a}e^{ax}\cos(bx+c) \Longrightarrow \quad aI - bJ = e^{ax}\cos(bx+c)\ \ldots(1)$

For $J = \int e^{ax}\sin(bx+c)\,dx$

Let $u = \sin(bx+c)$, $du = b\cos(bx+c)\,dx$, $dv = e^{ax}dx$, $v = \frac{1}{a}e^{ax}$

$J = \frac{1}{a}e^{ax}\sin(bx+c) - \int\left(\frac{1}{a}e^{ax}\right)[b\cos(bx+c)\,dx] = \frac{1}{a}e^{ax}\sin(bx+c) - \frac{b}{a}\int e^{ax}\cos(bx+c)\,dx$

$\frac{b}{a}I + J = \frac{1}{a}e^{ax}\sin(bx+c) \Longrightarrow bI + aJ = e^{ax}\sin(bx+c)\ \ldots(2)$

$a \times (1) + b \times (2)$, $(a^2+b^2)I = a\,e^{ax}\cos(bx+c) + b\,e^{ax}\sin(bx+c)$

$$\boldsymbol{I = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + C_1}$$

$a \times (2) - b \times (12)$, $(a^2+b^2)J = a\,e^{ax}\sin(bx+c) - b\,e^{ax}\cos(bx+c)$

$$\boldsymbol{J = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)] + C_2}$$

Method 2 (Differentiation)

$I = \int e^{ax}\cos(bx+c)\,dx$, $J = \int e^{ax} sin(bx+c)\,dx$

$\frac{d}{dx}[e^{ax}\cos(bx+c)] = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$

Integrate both sides, $a\int e^{ax}\cos(bx+c)\,dx - b\int e^{ax}\sin(bx+c)\,dx = e^{ax}\cos(bx+c)$

$aI - bJ = e^{ax}\cos(bx+c)\ \ldots(1)$

$\frac{d}{dx}[e^{ax} sin(bx+c)] = ae^{ax} sin(bx+c) + be^{ax}\cos(bx+c)$

Integrate both sides, $a\int e^{ax} sin(bx+c)dx + b\int e^{ax} cos(bx+c)dx = e^{ax} sin(bx+c)$

$bI + aJ = e^{ax} sin(bx+c)\ \ldots(2)$

Solving (1) and (2), we have $\boldsymbol{I = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + C_1}$

$$\boldsymbol{J = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)] + C_2}$$

Method 3 (Integration by parts)

It is more or less the same as Method 1, here instead we change u and v in the integration by parts and use a single equation instead of solving simultaneous equations.

$$I = \int e^{ax}\cos(bx+c)\,dx = \frac{1}{b}\int e^{ax} d[\sin(bx+c)] = \frac{1}{b}[e^{ax}\sin(bx+c) - \int \sin(bx+c) d(e^{ax})]$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) - \frac{a}{b}\int e^{ax}\sin(bx+c)\,dx = \frac{1}{b}e^{ax}\sin(bx+c) + \frac{a}{b^2}\int e^{ax}\, d[\cos(bx+c)]$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) + \frac{a}{b^2}[e^{ax}\cos(bx+c) - \int \cos(bx+c)\, d(e^{ax})]$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) + \frac{a}{b^2}e^{ax}\cos(bx+c) - \frac{a^2}{b^2}\int e^{ax}\cos(bx+c)\,dx$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) + \frac{a}{b^2}e^{ax}\cos(bx+c) - \frac{a^2}{b^2}I$$

$$I\left(1+\frac{a^2}{b^2}\right) = \frac{1}{b}e^{ax}\sin(bx+c) + \frac{a}{b^2}e^{ax}\cos(bx+c)$$

Solve for I, $\boldsymbol{I = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + C_1}$

Similarly, or more ingeniously differentiate the above equation partially with respect to c,

$$\boldsymbol{J = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)] + C_2}$$

Method 4 (Complex number)

$$I = \int e^{ax}\cos(bx+c)\,dx\,,\; J = \int e^{ax} sin(bx+c)\,dx$$

$$I + iJ = \int e^{ax}[\cos(bx+c) + i\;\sin(bx+c)]dx = \int e^{ax}e^{i(bx+c)}dx = \int e^{ax+i(bx+c)}dx$$

$$= \int e^{(a+bi)x+ic}dx = e^{ic}\int e^{(a+bi)x}dx = \frac{e^{ic}}{a+bi}e^{(a+bi)x} = \frac{e^{ax}}{a^2+b^2}(a-bi)e^{i(bx+c)}$$

$$= \frac{e^{ax}}{a^2+b^2}(a-bi)[\cos(bx+c) + i\;\sin(bx+c)]$$

Expand and compare real and imaginary parts, we have

$$\boldsymbol{I = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + C_1} \qquad \boldsymbol{J = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)] + C_2}$$

Method 5 (Differential equation)

$y = \int e^{ax}\cos(bx+c)\,dx \Longrightarrow y' = \int e^{ax}\cos(bx+c)\,dx$

The characteristic equation of the homogeneous ODE $y' = 0$ is $\lambda = 0$.

The complimentary solution is $y_C = C_1 e^{0\cdot x} = C_1$

The particular solution is $y_P = e^{ax}[A\cos(bx+c) + B\sin(bx+c)]$

Using the method of undetermined coefficients, we have

$y_P' = e^{ax}[-bA\sin(bx+c) + bB\cos(bx+c)\,] + ae^{ax}[A\cos(bx+c) + B\sin(bx+c)]$:

$= e^{ax}[(-bA + aB)\sin(bx+c) + (aA + bB)cos(bx+c)\,] \equiv e^{ax}\cos(bx+c)$

Hence $\begin{cases} -bA + aB = 0 \\ aA + bB = 1 \end{cases} \Longrightarrow \begin{cases} A = \frac{a}{a^2+b^2} \\ B = \frac{b}{a^2+b^2} \end{cases} \Longrightarrow y_P = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)]$

$\boldsymbol{y = y_P + y_C = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + C_1}$

Method 6

Take $a = r\cos\theta, b = rsin\ \theta \Longrightarrow r = \sqrt{a^2+b^2},\ \theta = tan^{-1}\left(\frac{b}{a}\right)$

The integrand

$$e^{ax}\cos(bx+c) = e^{ax}\cos\{[(bx+c) - \theta] + \theta\}$$

$$= e^{ax}\{\cos[(bx+c) - \theta]\cos\theta - \sin[(bx+c) - \theta]\sin\theta\}$$

$$= \frac{e^{ax}}{r}\{a\cos[(bx+c) - \theta] - b\sin[(bx+c) - \theta]\}$$

$$= \frac{d}{dx}\left\{\frac{e^{ax}}{r}\cos[(bx+c) - \theta]\right\}$$

$$\therefore \int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{r}\cos[(bx+c) - \theta] + C_1$$

$$= \frac{e^{ax}}{r}\{\cos(bx+c)\cos\theta + \sin(bx+c)\sin\theta\} + C_1$$

$$= \frac{e^{ax}}{r^2}\{a\cos(bx+c) + b\sin(bx+c)\} + C_1$$

$$\therefore \boldsymbol{I = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + C_1}$$